# MATH 1A − QUIZ 3 − SOLUTIONS

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(1) *(4 points)* Show, using the $\epsilon - \delta$ definition of a limit, that:

$$\lim_{x \to -4} 2x + 2 = -6$$

Let $f(x) = 2x + 2$

**Part I: Finding $\delta$**

1) $|f(x) - (-6)| = |2x + 2 + 6| = |2x + 8| = left|2(x + 4) = 2|x + 4|$
2) $2|x + 4| < \epsilon$ implies $|x + 4| < \frac{\epsilon}{2}$
3) Let $\boxed{\delta = \frac{\epsilon}{2}}$

**Part II: Showing your $\delta$ works**

1) Let $\epsilon > 0$ be given. Let $\delta = \frac{\epsilon}{2}$, and suppose $0 < |x + 4| < \delta$. Then $|x + 4| < \frac{\epsilon}{2}$
2) Then $|f(x) + 6| = 2|x + 4| < 2\frac{\epsilon}{2} = \epsilon$
3) Hence, if $0 < |x + 4| < \delta$, then $|f(x) - (-6)| < \epsilon$

*Date*: Friday, September 20th, 2013.

(2) *(4 points; 1 point each)* Evaluate the following limits or say the limit does not exist:

(a)

$$
\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\right)\left(\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\right)\\
&= \lim_{x\to 2}\frac{(\sqrt{6-x})^2-2^2}{\left(\sqrt{3-x}-1\right)\left(\sqrt{6-x}+2\right)}\\
&= \lim_{x\to 2}\frac{(6-x)-4}{\left(\sqrt{3-x}-1\right)\left(\sqrt{6-x}+2\right)}\\
&= \lim_{x\to 2}\left(\frac{2-x}{\sqrt{6-x}+2}\right)\left(\frac{1}{\sqrt{3-x}-1}\right)\\
&= \lim_{x\to 2}\left(\frac{2-x}{\sqrt{6-x}+2}\right)\left(\frac{1}{\sqrt{3-x}-1}\right)\left(\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)\\
&= \lim_{x\to 2}\left(\frac{2-x}{\sqrt{6-x}+2}\right)\left(\frac{\sqrt{3-x}+1}{(\sqrt{3-x})^2-1^2}\right)\\
&= \lim_{x\to 2}\left(\frac{2-x}{\sqrt{6-x}+2}\right)\left(\frac{\sqrt{3-x}+1}{(3-x)-1}\right)\\
&= \lim_{x\to 2}\left(\frac{\cancel{2-x}}{\sqrt{6-x}+2}\right)\left(\frac{\sqrt{3-x}+1}{\cancel{2-x}}\right)\\
&= \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\\
&=\frac{\sqrt{3-2}+1}{\sqrt{6-2}+2}\\
&=\frac{1+1}{2+2}\\
&=\frac{1}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\lim_{x\to 3^-}\frac{x^2-6x+9}{x^2-3x+2} &= \lim_{x\to 3^-}\frac{(x-3)^2}{(x-1)(x-2)}\\
&=\frac{(3^- -3)^2}{(3^- -1)(3^{-1}-2)}\\
&=\frac{(0^-)^2}{(2)(1)}\\
&=\frac{0^+}{2}\\
&=0
\end{aligned}
$$

(c) $\lim_{x\to 2} \frac{|x-3|+1}{|x-2|}$ ?

**Notice:** That as $x \to 2$, $x < 3$, so throughout, we have $|x-3| = -(x-3)$

$$\begin{aligned}
\lim_{x\to 2^+} \frac{|x-3|+1}{|x-2|} &= \lim_{x\to 2^+} \frac{-(x-3)+1}{x-2} \\
&= \lim_{x\to 2^+} \frac{4-x}{x-2} \\
&= \frac{4-2^+}{2^+-2} \\
&= \frac{2}{0^+} \\
&= \infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 2^-} \frac{|x-3|+1}{|x-2|} &= \lim_{x\to 2^-} \frac{-(x-3)+1}{-(x-2)} \\
&= \lim_{x\to 2^-} \frac{4-x}{2-x} \\
&= \frac{4-2^-}{2-2^-} \\
&= \frac{2}{0^+} \\
&= \infty
\end{aligned}$$

Hence, we get: $\lim_{x\to 2} \frac{|x-3|+1}{|x-2|} = \infty$

(d)

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x\sqrt{1+x}} - \frac{1}{x} &= \lim_{x\to 0} \frac{1-\sqrt{1+x}}{x\left(\sqrt{1+x}\right)} \\
&= \lim_{x\to 0} \frac{\left(1-\sqrt{1+x}\right)\left(1+\sqrt{1+x}\right)}{x\left(\sqrt{1+x}\right)\left(1+\sqrt{1+x}\right)} \\
&= \lim_{x\to 0} \frac{\left(1^2-\left(\sqrt{1+x}\right)^2\right)}{x\left(\sqrt{1+x}\right)\left(1+\sqrt{1+x}\right)} \\
&= \lim_{x\to 0} \frac{(1-(1+x))}{x\left(\sqrt{1+x}\right)\left(1+\sqrt{1+x}\right)} \\
&= \lim_{x\to 0} \frac{-\cancel{x}}{\cancel{x}\left(\sqrt{1+x}\right)\left(1+\sqrt{1+x}\right)} \\
&= \lim_{x\to 0} \frac{-1}{\sqrt{1+x}\left(1+\sqrt{1+x}\right)} \\
&= \frac{-1}{1(1+1)} \\
&= -\frac{1}{2}
\end{aligned}$$

(3) *(2 points)* Is the following function $f$ continuous at $0$? Why or why not? **Explain!**

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$$\begin{aligned} -1 \leq \sin\left(\frac{1}{x^2}\right) &\leq 1 \\ -x^2 \leq x^2 \sin\left(\frac{1}{x^2}\right) &\leq x^2 \\ -x^2 \leq f(x) &\leq x^2 \end{aligned}$$

Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, by **the squeeze theorem**, we have:

$$\lim_{x\to 0} f(x) = 0$$

However, because

$$\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$$

We get $\lim_{x\to 0} f(x) \neq f(0)$, and hence $f$ is **NOT** continuous at $0$.